Relation between quantum invariants of 3-manifolds and 2-dimensional CW-complexes

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Abstract

We show that the Reshetikhin-Turaev-Walker invariant of 3-manifolds can be normalized to obtain an invariant of 4-dimensional thickenings of 2-complexes. Moreover when the underlying semisimple tortile category comes from the Lie family ("quantum groups") over the ring $Z_{(p)}[v]$ where v is a primitive prime root of unity, the 0-term in the Ohtsuki expansion of this invariant depends only on the spine and is the Z/pZ invariant of 2-complexes defined in [10]. As a consequence it is shown that when the Euler characteristic is greater or equal to 1, the 2-complex invariant depends only on homology. The last statement doesn't hold for the negative Euler characteristic case.

1. Introduction. The construction of "quantum" invariants of 3-manifolds was introduced by Reshetikhin-Turaev [12] and put on a firmer footing by Walker [16], so we refer to these as "RTW" invariants. For elaborate complete developments see the books [14] and [8]. In general the input for this construction is a finite semisimple tortile category. Usually, however, the category is understood to be one of the Lie family obtained as subquotients of representations of a deformation of the universal enveloping algebra of a simple Lie algebra, specialize to a root of unity, see [9].

Analogous invariants of 2-dimensional CW complexes were introduced by the second author in [10] and it was called in [3] the Q-invariant, so we will continue using this name. In general the input for this construction is a finite semisimple *symmetric* monoidal category. Usually the category is taken to be one of the Lie family described by Gelfand-Kazhdan [5], obtained as subquotients of mod p representations of simple Lie algebras. These invariants are known to be invariant under deformations through 2-complexes. These deformations correspond to the Andrews-Curtis moves [1] for presentations of groups. The (generalized) Andrews-Curtis conjecture asserts that any simple homotopy equivalence of 2-complexes can be obtained by deformation through 2-complexes. This conjecture is expected to be false, and the Q- invariant was developed to try to detect counterexamples.

The present work gives a connection between these invariants in the standard setting (input categories from the Lie family) by relating them both to an invariant

of 4-dimensional thickenings of 2-complexes. More explicitly we fix a simple Lie algebra and a prime p greater than the dual Coxeter number of the algebra. Let $Z_{(p)}$ denote the integers localized at p (= rationals with denominators prime to p), and v be a p^{th} root of unity.

Theorem 1.1 *With Lie algebra and prime as above, there is $\hat{Z}(W) \in Z_{(p)}[v]$ defined for W a 4-dimensional thickening of a 2-complex such that*

(a) $\hat{Z}(W)$ is invariant under deformations through such thickenings;

(b) there is a normalization of $\hat{Z}(W)$ in $Q[v]$ giving the RTW invariant of ∂W; and

(c) the reduction mod p of $\hat{Z}(W)$ is the Q- invariant of the spine of W.

The normalization used in (b) and the proof of the theorem are given in 3.12.

Part (c) provides topological interpretations for (at least some) "Ohtsuki expansions" of quantum invariants [11]. In $Z_{(p)}[v]$ we can write

$$\hat{Z}(W) = a_0 + a_1(1-v) + \ldots + a_{p-1}(v-1)^{(p-1)}$$

with $a_i \in Z_{(p)}$. The mod p reductions of the coefficients are well-defined, and are the Ohtsuki expansion of $\hat{Z}(W)$. $(1-v)$ is trivial in the mod p reduction of $Z_{(p)}[v]$ so the mod p reduction of $\hat{Z}$ is equal to the mod p reduction of a_0. In other words, the 0 term in the Ohtsuki expansion of $\hat{Z}(W)$ is the Q- invariant of the spine of W.

The theorem also has implications for the Q- invariants.

Corollary 1.2 *When the Euler characteristic of the 2-complex is greater or equal to 1 its quantum invariant is determined by the homology of the complex. In particular, when the invariant is defined using a Gelfand-Kazhdan category it vanishes when the second homology is nonzero, and otherwise is determined by the torsion subgroup of H_1.*

The corollary implies that the invariant cannot detect counterexamples to the original Andrews-Curtis conjecture which concerns contractable complexes. This result was strongly suggested by numerical studies of the invariant (described at *http://www.math.vt.edu/quantum_topology*), but of course could not be proved that way.

In section 4 we use the corollary and computations for cyclic presentations to get an explicit formula for invariants defined using the simplest Lie algebra. To state this recall b_2 (the second Betti number) is the rank of $H_2(X; Z)$, and t_1 is the order of the torsion subgroup of $H_1(X; Z)$.

Proposition 1.3 *Suppose X is a 2-complex with Euler characteristic greater or equal to 1. Then the class-0 $SL(2)$ Q- invariant of X is 0 if $b_2 > 0$ or if p divides t_1, and is $t_1^{-2} \in Z/pZ$ otherwise.*

Note the inverse in the second case is to be taken in Z/pZ. Explicitly, the invariant is the mod p reduction of r^2, where $rt_1 \equiv 1 \bmod p$.

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2. Preliminaries and notations

2.4 Let R be the ring of integers localized at p, with a p^{th} root of unity adjoined, so

$$R = Z_{(p)}[v]/ < 1 + v + v^2 + ... + v^{(p-1)} > .$$

Let $\mathcal{A}$ be a semisimple tortile category (see [13]) over R and let $S = \{1, a, b, \ldots\}$ be a set of chosen representatives for the simple objects in $\mathcal{A}$, i.e. $\mathbf{1}$ denotes the identity object and small letters indicate simple objects. A^* indicates the dual object of A, and A^{**} is canonically identified with A. Furthermore, we use the following notations:

$$\begin{array}{rl}
 & \mathcal{A} \times \mathcal{A} \to \mathcal{A} \quad \text{(product)}; \\
\alpha_{A,B,C} : & (AB)C \to A(BC) \quad \text{(associativity)}; \\
\gamma_{A,B} : & AB \to BA, \text{(commutativity)}; \\
\Lambda_A : & \mathbf{1} \to A^* \diamond A \text{ (coform)}; \\
\lambda_A : & A \diamond A^* \to \mathbf{1} \text{ (form)}; \\
r_A = \lambda_A \Lambda_{A^*} : & \mathbf{1} \to \mathbf{1} \text{ (rank)}; \\
\theta_A : & A \to A\,\mathbf{1} \xrightarrow{\Lambda_{A^*}} A(AA^*) \to (AA)A^* \\
 & \xrightarrow{\gamma_{A,A}} (AA)A^* \to A(AA^*) \xrightarrow{\lambda_A} A\,\mathbf{1} \to A \text{ (twist)} \\
dim(A,B) & \text{is the dimension of } hom(A,B).
\end{array}$$

The main examples come from Lie algebras. Lustig [9] defines a version of the quantum enveloping algebra specialized at the root of unity v as an algebra over $Z[v]$. Gelfand-Kazhdan [5] define a subquotient category of the representations of this algebra. They show that when reduced mod p the tensor product of representations induces a symmetric monoidal structure on the subquotient. It follows from this that tensor product induces a monoidal structure (tortile, but no longer fully symmetric) on the subquotient with $Z_{(p)}[v]$ coefficients. The point to check is that associativity maps are isomorphisms. But the hom sets are finitely generated projective modules so mod p isomorphism implies isomorphism over $Z_{(p)}[v]$. This conclusion can be considerably refined. General finiteness considerations show the structure becomes monoidal over $Z[\frac{1}{Q}, v]$ where Q is some finite set of primes. We conjecture it is sufficient to invert the primes less than the dual Coxeter number of the algebra.

The semisimplicity of the category implies that given $a, b \in S$ there are bases for spaces of morphisms to or from the product. For every simple object z choose bases $inj_k(z, ab) : z \to ab$, and $proj_k(ab, z) : ab \to z$, $1 \le k \le dim(z, ab)$, such that:

$$proj_l(ab, z) \circ inj_k(z, ab) = \delta_{k,l} id_z;$$
$$\sum_{z,k} inj_k(z, ab) \circ proj_k(ab, z) = id_{ab}.$$

Using the coherence results in [13] we can represent certain morphisms in $\mathcal{A}$ by labeled tangle diagrams (with blackboard framing). Let $\underline{\mathrm{A}} = (A_1,\, A_2 \, \ldots, A_k)$ and $\underline{\mathrm{B}} = (B_1,\, B_2 \, \ldots, B_l)$ be sequences of objects in $\mathcal{A}$. Chose bracketings $\mathcal{B}_1$ and $\mathcal{B}_2$ of the sequences to specify a way to form the products. Then there is a one to one correspondence between the set of equivalence classes of labeled tangle diagrams $D : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$ and the set of morphisms $\mathcal{B}_1(\underline{\mathrm{A}}) \to \mathcal{B}_2(\underline{\mathrm{B}})$ in $\mathcal{A}$ formed by composition of products of the elementary morphisms α, γ, λ, Λ and identities. The correspondence is determined by the images of the elementary morphisms as shown in figure 1.

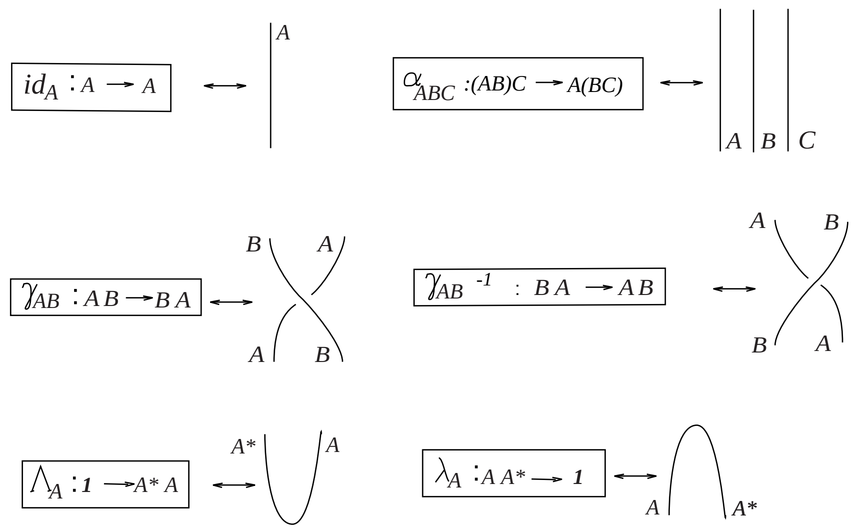
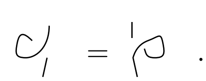


Figure 1: Elementary diagrams.

The equivalence relation on diagrams is generated by isotopy in the plane and modified Reidemeiser moves, where the first move (involving the twist) is replaced by



We will use labels that are "linear combinations" of objects with coefficients in R, and associate morphisms to these as follows. The morphisms defined with

genuine object labels are additive in each label, so

$$\mathrm{morph}(\cdots, A+B, \cdots) = \mathrm{morph}(\cdots, A, \cdots) + \mathrm{morph}(\cdots, B, \cdots).$$

Therefore if $r, s \in R$ we can define

$$\mathrm{morph}(\cdots, rA+sB, \cdots) = r\,\mathrm{morph}(\cdots, A, \cdots) + s\,\mathrm{morph}(\cdots, B, \cdots).$$

In particular we use the "universal" label $U = \sum_{a \in S} r_a a$ where simple objects are weighted by their ranks. This is different from the normalization used in other constructions, and is done to keep the constructions in the ring R as long as possible.

2.5 A semisimple tortile category $\mathcal{A}$ is called *nondegenerate* if the following three conditions are satisfied:

(a) $X^2 = \sum_{a \in \Sigma} {r_a}^2$ is not a zero-divisor;

(b) $C_+ = \sum_{a \in \Sigma} {r_a}^2 \theta_a$ and $C_- = \sum_{a \in \Sigma} {r_a}^2 {\theta_a}^{-1}$ are not zero-divisors;

(c) For any $a \in \Sigma$, $a \neq \mathbf{1}$, there exist labeled tangle diagrams D and D' such that each of them contains a closed tangle component labeled by a. Moreover, D and D' are equal if this closed component is deleted, but the morphisms in $\mathcal{A}$ corresponding to D and D' are different.

The constructions of [8, 12, 14, 16] give 3-manifold invariants from nondegenerate tortile categories. These invariants take values in the ring with inverses for the elements in (a) and (b) adjoined, not necessarily in the original ring.

The Lie family of categories are nondegenerate in this sense. These categories also satisfy:

(d) θ_a for a simple object a acts as $v^{t(a)} id_a$ for some integer $t(a)$.

From now on we assume that $\mathcal{A}$ is a semisimple tortile category over $R = Z_{(p)}[v]$ satisfying the conditions (a)–(d).

2.6 We list two properties of nondegenerate categories, which show how special such categories are (for the proofs see [16]).

The first property states the invariance under a band-connected sum or difference of one tangle components with another closed component. Let a tangle diagram D contain a closed component K labeled with the universal label $U = \sum_{a \in S} r_a a$, and D' is obtained from D by sliding any other tangle component along K. Then if $\mathcal{A}$ satisfies (a) and (b), the morphisms corresponding to the original tangle diagram and the new one are the same.

The second property requires that the three conditions (a)–(c) are satisfied and describes the morphism corresponding to the diagram

$$T(b_1, b_2 \ldots, b_k) : (b_1, b_2 \ldots, b_k) \to (b_1, b_2 \ldots, b_k)$$

consisting from k straight segments labeled by the b_i's and a unknoted closed component which goes around them, labeled by U. The diagram is shown in figure 2. It is shown in [16] that

$$T(b) = \begin{cases} X^2\, id_1, & \text{if } b = \mathbf{1}; \\ 0, & \text{if } b \neq \mathbf{1}. \end{cases}$$

Then from the semisimplicity of the category it follows that

$$T(b_1, b_2 \ldots, b_k) = X^2 \sum_{i=1}^{dim(\mathbf{1}, b_1 b_2 \ldots b_k)} inj_i(1, b_1 b_2 \ldots b_k) \circ proj_i(b_1 b_2 \ldots b_k, \mathbf{1}).$$

This shows $\frac{1}{X^2} T(b_1, b_2 \ldots, b_k)$ is a morphism in the category (i.e. over R, in spite of division by X^2). More precisely it is the trace of the projection over the trivial summand.

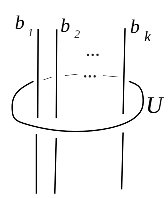


Figure 2: The projection morphism $T(b_1, b_2 \ldots, b_k)$.

From the evaluation of T above after doing a handle slide, it follows that

$$C_+ C_- = X^2.$$

2.7 Let $\phi_p : R \to Z_p$ denote reduction mod p. Explicitly this is given by

$$\phi_p(a_0 + a_1(1-v) + \ldots a_l(1-v)^l) = a_0 \text{ mod } p.$$

The following explains the role of hypothesis 2.5(d) in the relationship between the Lie type tortile categories over R and the Gelfand-Kazhdan categories over Z/pZ.

Proposition 2.8 *Let $\mathcal{A}$ be a semisimple tortile category over R satisfying condition 2.5(d) and let $\phi_p(\mathcal{A})$ be the category over Z/pZ having the same objects as $\mathcal{A}$, and morphism modules the mod p reductions of morphisms in $\mathcal{A}$. Then $\phi_p(\mathcal{A})$ is finite semisimple and symmetric.*

It is clear that mod p reduction of structure in $\mathcal{A}$ gives a finite semisimple tortile category. The twists and commutativities satisfy

$$\theta_{ab} = \theta_a^{-1}\theta_b^{-1}\gamma_{b,a}\gamma_{a,b}.$$

For the quotient to be symmetric we need $\gamma_{b,a} \equiv \gamma_{a,b}^{-1}$ mod p, or more precisely the mod p reductions of the twists θ_A should all be identities. Since θ is additive it is sufficient to show this for simple objects. Hypothesis 2.5(d) asserts that these are powers of v times identities. ϕ_p takes powers of v to 1, so the mod p reductions are identity morphisms as required.

3. Proof of the Main Result.

3.9 RTW Invariants Let M be a closed 3-manifold, obtained via surgery on a framed link L. Define $Z(L)$ to be the morphism $\mathbf{1} \to \mathbf{1}$ in $\mathcal{A}$ induced by the diagram with underlying framed link L and label U on every component. Then the quantum invariant of M is the element of $R[\frac{1}{X}, \frac{1}{C_-}, \frac{1}{C_+}]$ defined by

$$Z_{RTW}(M) = \frac{1}{C_+^{\sigma_+}C_-^{\sigma_-}X^{\sigma_0}}Z(L)$$

where σ_+, σ_- and σ_0 are the numbers of positive, negative and zero eigenvalues of the linking matrix of L.

The unnormalized $Z(L)$ is a link invariant, and is invariant under the second Kirby move (band-connected sum or difference of two link components). It does not define a manifold invariant because it is not invariant under the first Kirby move (adding or deleting an unknot of framing ± 1 away of the rest of the link). The normalization factor in $Z_{RTW}(M)$ compensates for this non-invariance, so $Z_{RTW}(M)$ is a manifold invariant. Note the normalized invariant takes values in a larger ring because (in our cases) $C_\pm$ and X are not invertible in R.

3.10 4-thickening invariants A 4-thickening W of a 2 CW-complex P is a 4-dimensional manifold together with a decomposition as a handlebody with 0-, 1-, and 2-handles and an identification of the spine of the handlebody structure with P. Two 4-thickenings will be called 2-equivalent if they can be deformed into each other by a 2-deformation, i.e. by 1- and 2-handle slides and 0-1 and 1-2 handle cancellations or introductions. We observe that a 2-equivalence of 4-thickenings induces a 2-equivalence (in the 2-complex sense) of their spines.

A 4-thickening with a single 0-handle can be described by a framed link in S^3, obtained from the attaching maps of the 1- and 2-handles. The 1- handles are being represented by dotted unknots of framing 0 (the unknot represents the attaching map of a canceling 2-handle), and the 2- handles correspond to the undotted components. An example (using the blackboard framing) is shown in figure 3. Then ([7]) two 4-thickenings are 2-equivalent if and only if the corresponding framed links can be deformed into each other through:

(a) isotopy of framed links;

(b) handle moves

i) band-connected sum or difference of two dotted link components (sliding an 1-handle over another 1-handle);

ii) band-connected sum or difference of two undotted link components (sliding a 2-handle over a 2-handle);

iii) band-connected sum or difference of one undotted link component with one dotted link component (sliding a 2-handle over 1-handle);

(c) any pair of one dotted component C and one undotted component D can be removed or added if the geometric intersection number of D and the Seifert surface of C is ± 1 (1-2 handle cancellation or introduction).

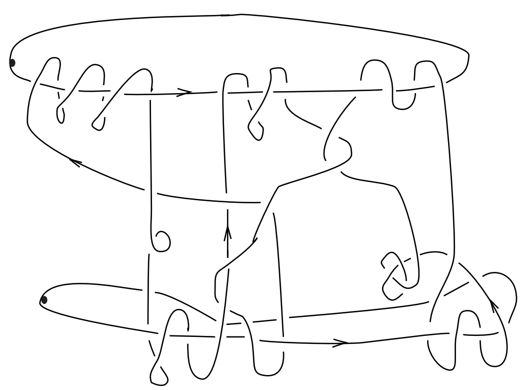


Figure 3: A 4-thickening with spine described by the presentation $P = < x, y | x^3y^2x^2y^{-1}, x^{-2}y^2 >$.

Suppose L is a framed link diagram describing a 4-thickening. The link invariant $Z(L)$ is unchanged by moves (a) and (b) but is not invariant under (c). For one thing we have not distinguished between dotted and undotted components, which amounts to exchanging all 1-handles with their cancelling 2-handles. The RTW normalization accepts this exchange and normalizes $Z(L)$ to be invariant under the first Kirby move. Here we use a different normalization that records something about the 1-handles.

Proposition 3.11 *Let W be a 4-thickening represented by a framed link L with n dotted components. Then*

$$\hat{Z}(W) = \frac{1}{X^{2n}} Z(L)$$

is invariant under a 2-deformation.

The only thing to check is the invariance under (c). Move (c) corresponds to changing L into L' by introducing a dotted unknot C and an undotted link component K such that the geometric intersection number of K and the disk bound by C is ± 1. According to the second property of nondegenerate categories in 2.6, C can be removed, and the label of K changed to the trivial label. But a component with the trivial label doesn't contribute to $\hat{Z}(L')$, i.e.

$$Z(L') = X^2\, Z(L).$$

3.12 Proof of 1.1 Let $\mathcal{A}$ be a tortile category over R satisfying the nondegeneracy conditions 2.5(a)–(d), and let W be a 4-thickening of a 2-complex P. Then to prove 1.1 we show

(a) $\hat{Z}(W) \in R$;

(b) the mod p reduction $\phi_p(\hat{Z}(W)) = Z_Q(P)$ is a 2-deformation invariant of the spine P.

Proof of (a). Let L be a link describing W, and let n be the number of dotted components of L. $\hat{Z}(W) = \frac{1}{X^{2n}} Z(L)$ and $Z(L) \in R$, so the only problem comes from the factor $\frac{1}{X^{2n}}$ (recall X is not invertible in R). But the diagram for L can be deformed into a composition of tangle diagrams, such that n of the composition factors are of the form $L'_k \otimes T(a_{i_1}, a_{i_2}, \ldots, a_{i_{s(k)}}) \otimes L''_k$, where $1 \le k \le n$, and $s(k)$ is the sum of the absolute values of all the exponents of the k'th generator in all relations. Here we are regarding the 2-complex P as a presentation, with generators the 1-cells and relations the 2-cells. Recall (2.6) that $\frac{1}{X^2} T(a_{i_1}, a_{i_2}, \ldots, a_{i_{s(k)}})$ is in R. Since there are n of these factors we see that we can divide $Z(L)$ by X^{2n} in R.

Proof of (b). To prove (b) it is sufficient to identify the mod p reduction as the Q- invariant, since this is known to be a 2-complex invariant. However we use a different approach that also gives a proof of 1.2. In this approach we define Z_Q by lifting: applying the 4-thickening invariant to a canonical thickening of the spine, then reducing mod p.

Suppose P is a presentation. This means P is a 2-complex with a single 0-cell, an order and orientations for the 1-cells, and attaching maps for the 2-cells expressed as words in the 1-cells. Then there is a standard thickening W_P defined by Huck [6]. This thickening is described by a framed link L_P with $n+m$ components, such that any component is an unknot of framing 0 and the geometric intersection number of any two undotted or any two dotted components is 0. The precise definition of L_0 can be given given as a closure of a braid. Let B_{n+m} be the braid group on $n+m$ strings. Let y_j, $j = 2 \ldots n+m$ be the braid group generator corresponding to interchanging the places of the $j-1$ and the j string. Then let $r_{j,k}$, $j < k$ be the group element which moves the j string to the k place:

$$r_{j,k} = y_{j+1} y_{j+2} \cdots y_{k-1} y_k.$$

For each $j = 1 \ldots m$ we define a homomorphism ψ_j from the free group F_n on n generators $x_1, x_2, \ldots, x_n$ into B_{n+m} such that

$$\psi_j(x_k) = r_{j,k-1} y^2_{k+m} (r_{j,k-1})^{-1},$$

where we assume $r_{ii} = 1$. The link L_P is defined to be the closure of the braid:

$$\hat{P} = \psi_1(R_1)\psi_2(R_2)\ldots\psi_m(R_m).$$

An example is given in figure 4.

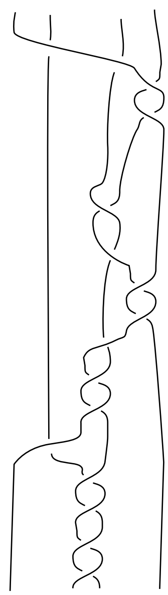


Figure 4: The braid $\hat{P}$ for the 2-complex $P =< x, y | x^2, xyx^{-1}y^{-1} >$.

For the purposes here we define $Z_Q(P) = \phi_p(\hat{Z}(W_P))$.

Next we show this is the mod p reduction of the general $\hat{Z}$ invariant. First observe that if L is the link describing an arbitrary thickening W of P, its diagram can be changed into the one for W_P by (1) a finite number of crossings not involving dotted components; (2) adding twists on undotted components; and (3) Reidemeister's moves. But it follows from from 2.7 it follows that $\phi_p(\hat{Z}(W))$ is unchanged under these operations, i.e., $\phi_p(\hat{Z}(W)) = \phi_p(\hat{Z}(W_P))$. This is true because the diagram can be sliced so that the changes are performed in slices not containing a dotted component. Crossing involving a dotted component must be avoided because this takes the invariant out of the ring R.

Finally we show that $Z_Q(P)$, as defined here, is invariant under a 2-deformation. For this it is enough to see that for a single 2-deformation move $f : P \to P'$ there exist thickenings W of P and W' of P', and a 2-deformation of thickenings

$F : W \to W'$ that realizes the move on the spines. The various sliding moves, and addition of a cancelling 1-2 pair of handles can be done starting with any W, and we let W' be the result. To remove a cancelling pair of handles we begin with some W' and let W be the thickening obtained by introducing handles.

3.13 Proof of 1.2 Suppose P is a presentation. Let L be the framed link constructed above to define the thickening W_P. L has the feature that the undotted components form a trivial framed link. There is a "duality" defined for such framed links by interchanging the dotted and undotted sublinks. Geometrically this corresponds to replacing 1-handles by their cancelling 2-handles, and replacing 2-handles with a particular set of cancelling 1-handles. Denote the spine of the associated handlebody by P^*, and the accisiated handlebody with $(W_P)^*$. Huck [6] observes that there is an embedding of W_P in S^4 with complement $(W_P)^*$.

If P is given in terms of generators and relations by

$$P =< x_1, x_2, \dots, x_n | \; R_1, R_2, \dots, R_m >$$

then the dual presentation can be described explicitly as

$$P^* =< r_1, r_2, \dots, r_m | \; X_1, X_2, \dots, X_n > .$$

The relations are

$$X_k = r_1^{f_k^1} r_2^{f_k^2} \dots r_m^{f_k^m} .$$

where f_k^l is the total exponent of x_k in R_l. Recall that the total exponents are entries in the boundary homomorphism in the cellular chain complex. Since the chain complex is characterized up to deformation by the homology, it follows that the dual presentation (up to a 2-deformation) depends only on the homology of P.

Recall that $\hat{Z}$ is defined by normalizing a link invariant. Specifically

$$\hat{Z}(W) = \frac{1}{X^{2n}} Z(L)$$

where n is the number of 1-handles (dotted components) of W and L is the link obtained by forgetting dots. Since W_P and $(W_P)^*$ have the same underlying link, and n and m 1-handles respectively, we get

$$X^{2n} \, \hat{Z}(W_P) = X^{2m} \, \hat{Z}((W_P)^*).$$

In particular, if $m \geq n$, $Z_Q(P) = \phi_p(X^{2(m-n)}) Z_Q(P^*)$. Since P^* depends only on the homology of P, it follows that in this case, the invariant depends only on homology.

To complete the proof of the corollary we need to refine this in terms of Betti numbers and torsion in the case that the invariant is defined using a Gelfand-Kazhdan category. The important point is that for such categories $\phi_p(X^2) = 0$.

Recall b_i is defined by $H_i(P;Z) \simeq Z^{b_i} \oplus$(torsion). Let P^{ab} be the abelianization of P. From above it follows that $Z_Q(P) = Z_Q(P^{ab})$. Moreover, there is a 2-deformation

$$P^{ab} \to (\vee^{b_1} S^1) \vee T \vee (\vee^{b_2} S^2),$$

where T has torsion homology and $\vee$ denotes 1-point union. Since Z_Q is multiplicative under 1-point unions the corollary reduces to showing $Z_Q(S^1) = 1$ and $Z_Q(S^2) = 0$. The second computation can be done formally. Notice these are dual in the sense that if $P = S^1$, which as a presentation has a single generator and no relations, then the dual has no generators and one relation, so is S^2. The relation between dual presentations gives

$$X^2\, \hat{Z}(W_{S^1}) = \hat{Z}(W_{S^2}).$$

Hence $Z_Q(S^2) = 0$. More directly these are both defined in terms of the link invariant $Z(L)$ where L is a single unlinked circle. It is a simple computation that $Z(L) = X^2$. The invariant of S^1 is obtained by dividing this by X^2 and reducing mod p, so $Z_Q(S^1) = 1$ as required.

3.14 The above corollary is not true in the case when the Euler characteristic of the complex is smaller or equal to 0. The simplest example comes from comparing the invariant of the 2-complexes $P =< x,y|\ xyx^{-1}y^{-1} >$ and $P' =< x,y|\ \emptyset >$, where $\emptyset$ denotes the trivial relation. The invariant of P' is 0 (the link diagram of $W_{P'}$ consists of one undotted unknot and two dotted unknots, all disjoint from each other). The invariant of P, instead, is equal to the cardinality $|S|$ of the set of simple objects S. The last asertion is proven in [3] but can also be seen using the present framework. In fact figure 5 shows that

$$\hat{Z}(W_P) = \sum_{b \in S} v^{t(b)}.$$

Hence $Z_Q(P) = \phi_p(\hat{Z}(W_P)) = |S|$.

4. Evaluation of the $SL(2)$ 2-complex invariant.

4.15 Here we give the proof of Proposition 1.3, explicitly describing the 2-complex invariant for a simple class of categories. According to 1.2 when the Euler characteristic is greater or equal to 1, the invariant depends only on the homology of the complex. Given such 2-complex there is one with the same homology which is a 1-point union of copies of S^1, S^2, and cyclic presentations $< x \mid x^n >$ for n a power of a prime. The invariant is multiplicative under 1-point unions and we have already found the values for S^1 and S^2. The proposition is therefore reduced to proving the cyclic case:

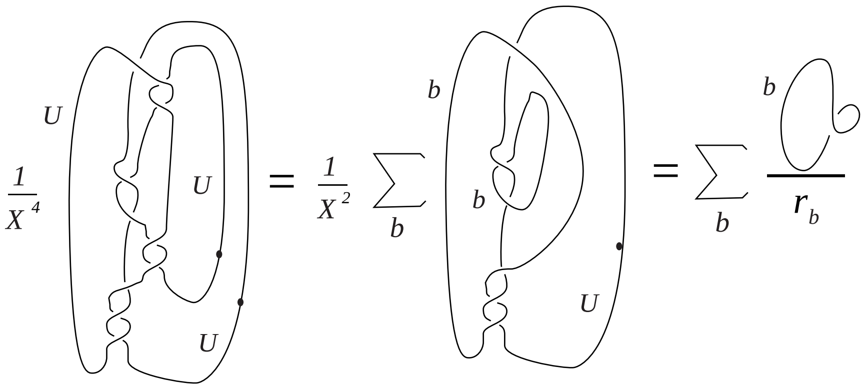


Figure 5: Evaluation of $\hat{Z}(W_P)$ for $P =< x, y|\ xyx^{-1}y^{-1} >$.

Lemma 4.16 *Let $\mathcal{A}$ be the category obtained from class-0 $SL(2)$ representations at a p^{th} root of unity, $p \geq 5$. Then*

$$Z_Q(< x \mid x^n >) = \begin{cases} 0 & \textit{if } p \textit{ divides } n; \\ n^{-2} \in Z/pZ, & \textit{otherwise.} \end{cases}$$

The category has simple objects the simple representations of the quantum enveloping algebra of $SL(2)$ with highest weights w satisfying

(i) w is even;

(ii) $0 \leq w \leq p - 3$.

The formulas here are taken from [2], where the definition of quantum enveloping algebra is the one of Lusztig, introduced in [9]. This is slightly different from the definition used in [12] and the literature following it, and this leads to slight differences in the formulas.

4.17 Let $w = 2z$ be a weight satisfying the conditions above. Then for the rank and the twist of the representation with highest weight w, we have:

$$r_w = [2z + 1], \quad \text{and} \quad \theta_w = v^{-2z(z+1)},$$

where $[n] = \frac{v^n - v^{-n}}{v^1 - v^{-1}}$. $[n]$ is often called a “quantum integer.”

For completeness we list formulas from [17] which are used below. Let

$$g_1 = \sum_{z=0}^{p-1} v^{z^2} = \sqcap_{k=1}^{\frac{p-1}{2}} (v^{2k-1} - v^{-2k+1}) = (v - v^{-1})^{\frac{p-1}{2}} \sqcap_{k=1}^{\frac{p-1}{2}} [2k + 1] \in (v - 1)^{\frac{p-1}{2}} k_p,$$

denote the Gauss sum, and

$$\left(\frac{a}{p}\right) = \begin{cases} 0 \quad \text{if } a = 0 \quad mod \quad p, \\ 1 \quad \text{if } a \text{ is a square } mod \quad p, \\ -1 \quad \text{if } a \text{ is not a square } mod \quad p, \end{cases}$$

be the Legendre symbol. Then in R, $p = (-1)^{\frac{p-1}{2}} g_1^2 = (v - v^{-1})^{p-1} \sqcap_{k=1}^{\frac{p-1}{2}} [2k+1]^2$. For the present categories, using *Mathematica* and the formula above, we obtain that

$$X^2 = \frac{-p}{(v - v^{-1})^2} = \frac{(-1)^{\frac{p+1}{2}}}{(v - v^{-1})^2} g_1^2 \in (v-1)^{p-3} k_p.$$

Note this shows the mod p reduction vanishes, as mentioned above.

4.18 According to 1.1, the invariant of the cyclic 2-complex is the mod p reduction of a normalized link invariant:

$$Z_Q(< x \mid x^n >) = \phi_p(\hat{Z}(L_n)) = \phi_p(\frac{1}{X^2} Z(L_n)),$$

where the link L_n is shown in figure 6 (a). This can be transformed by a handle slide to the link in figure 6 (b). This shows the link invariant is a product with factors corresponding to the two components, namely $F(n)F(-n))$, where

$$F(n) = \sum_{z=1}^{\frac{p-3}{2}} (r_{2z})^2 (\theta_{2z})^n = \sum_{z=1}^{\frac{p-3}{2}} [2z+1]^2 v^{-2zn(z+1)}.$$

If n is divisible by p then $F(n) = F(-n) = X^2$, so the 2-complex invariant is the mod p reduction of X^2 and therefore vanishes.

We now proceed with values of n not divisible by p. First observe that the 3-manifold associated to the link in 6 (b) is the connected sum of two Lens spaces. Quantum invariants of Lens spaces, as can be seen from the formula for $F(n)$, are reduced to calculating Gauss sums, and have been studied for example in [15]. Since the category here is somewhat different the calculation needs to be redone, but in a similar way we obtain

$$F(n) = \left(\frac{-n/2}{p}\right) \frac{g_1 v^{\frac{n^2+1}{2n}}}{(v - v^{-1})} [\bar{n}],$$

where $\bar{n}$ denotes the inverse of n in Z/pZ. Putting this in the expression for $Z_Q(< x \mid x^n >)$ and reducing gives $\bar{n}^2$, as required.

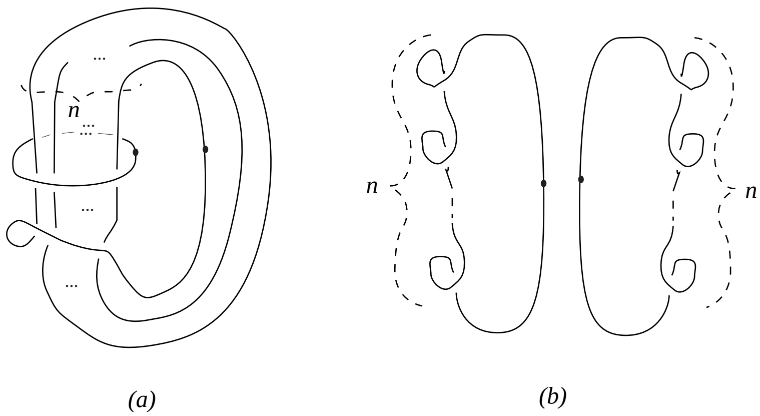


Figure 6: The diagram corresponding to the cyclic group.

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